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P. Dheena and C. Jenila

Abstract. In this paper we introduce the notion of NI near-rings similar to the notion introduced in rings. We give topological properties of collection of strongly prime ideals in NI near-rings. We have shown that if N is a NI and weakly pm near-ring, then $Max(N)$ is a compact Hausdorff space. We have also shown that if N is a NI near-ring, then for every $a \in N$, $cl(D(a)) = V(N^*(N)_a) = Supp(a) = SSpec(N) \backslash int\ V(a)$.

1. Introduction

Throughout this paper, N stands for a zero-symmetric near-ring with identity and all prime ideals of N are assumed to be proper. We use $P(N)$, $N^*(N)$ and $N(N)$ to represent the prime radical, the nilradical (i.e., the sum of all nil ideals) and the set of all nilpotent elements of N, respectively. An ideal P of N is prime if for any two ideals A and B of N, $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$. An ideal P of N is said to be completely prime if $ab \in P$ implies $a \in P$ or $b \in P$ for any $a, b \in N$. An ideal S of N is said to be completely semiprime if $a^2 \in S$ implies $a \in S$ for any $a \in N$.

An ideal P of N is said to be strongly prime if P is prime and N/P has no non-zero nil ideals. A near-ring N is said to be strongly prime if the ideal $\{0\}$ is strongly prime. An ideal P of a near-ring is minimal strongly prime ideal if P is minimal among strongly prime ideals of N. Observe that every completely prime ideal of N is strongly prime and every strongly prime ideal is prime but the converses do not hold.

Note that $N^*(N)$ of a near-ring N is the unique maximal nil ideal of N. For a near-ring N, $N^*(N) = \cap\{P \mid P$ is a strongly prime ideal of $N\} = \cap\{P \mid P$ is a minimal strongly prime ideal of $N\}$ by ([2], Lemma 1.5).

A near-ring is called reduced if it has no nonzero nilpotent elements. Now we introduce the notion of NI near-rings. A near-ring N is called NI if $N^*(N) = N(N)$. Note that N is NI if and only if $N(N)$ forms an ideal if and only if

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$N/N^*(N)$ is reduced. Topological properties of strongly prime ideals of NI rings have been characterized in [3]. Dheena and Sivakumar [2] have obtained only the properties of NI near-rings. Now we obtain topological properties of strongly prime ideals of NI near-rings. In this paper we study the structure of NI near-rings relating to strongly prime ideals and we associate the near-ring properties and topological properties. We extend the results obtained by Hwang et al. [3] for NI rings to NI near-rings. A near-ring N is called 2-primal if $P(N) = N(N)$. Clearly 2-primal near-rings are NI but the converse need not hold.

We use $SSpec(N)$ and $Max(N)$ for the space of all strongly prime ideals and the subspace of all maximal ideals of N, respectively. For any $a \in N$, we define $V(a) = \{P \in SSpec(N) \mid a \in P\}$ and $D(a) = SSpec(N)\backslash V(a)$. Let $V(J) = \cap_{a\in J} V(a)$, where J is an ideal of N. Then $F = \{V(J) \mid J$ is ideal of $N\}$ is closed under finite union and arbitrary intersections, so that there is a topology on $SSpec(N)$ for which F is the family of closed sets. This is called the Zariski topology (see [9]). For any subset A of N, $\langle A\rangle$ denotes the ideal of N generated by A. For any $a \in N$, $\langle a\rangle$ stands for the ideal of N generated by a. Note that $V(A) = V(\langle A\rangle)$ for any subset A of N. Let $\mathscr{B} = \{D(a) \mid a \in N\}$. Then $\mathscr{B}$ is a basis for a topology of $SSpec(N)$.

The operations cl and int denote the closure and the interior in $SSpec(N)$. For any subset S of N, we define $N^*(N)_S = \{n \in N \mid nS \subseteq N^*(N)\}$. We set $Supp(a) = \cap_{x\in N^*(N)_a} V(x)$. In this paper the notations of near-ring are from [8] and the notations of topology are from [7].

2. Preliminaries

Following Lambek [4], we have the following definition for symmetric near-ring.

Definition 2.1. A near-ring N is called symmetric if $abc = 0$ implies $acb = 0$ for all $a, b, c \in N$.

Note that N is symmetric if and only if $a_1a_2\cdots a_n = 0$, with n any positive integer, implies $a_{\sigma(1)}a_{\sigma(2)}\cdots a_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \ldots, n\}$ and $a_i \in N$.

We first need the following lemmas.

Lemma 2.2. *For a near-ring N the following conditions are equivalent*:

(1) *N is NI.*
(2) *Every minimal strongly prime ideal of N is completely prime.*
(3) *$N/N^*(N)$ is a subdirect product of domains.*
(4) *$N/N^*(N)$ is a reduced near-ring.*
(5) *$N/N^*(N)$ is a symmetric near-ring.*

Proof. (1) $\Leftrightarrow$ (2) is proved by Dheena and Sivakumar ([2], Theorem 2.6). The other implications are straightforward. ☐

Lemma 2.3 ([2], Theorem 1.4). *If M is a multiplicative subset in $N\backslash 0$, then there exists a strongly prime ideal P of N such that $P \cap M = \phi$.*

3. Topological space of $SSpec(N)$

In this section, we associate the near-ring properties of N and the topological properties of $SSpec(N)$. A near-ring N is called (weakly) pm if for each (strongly) prime ideal P of N, there exists unique maximal ideal M of N such that $P \subseteq M$. Clearly pm near-rings are weakly pm near-rings. Let A be a subset of N. We denote the lattice of all ideals of N by $Idl(N)$ and $S(A) = \bigcap_{A \subseteq P} P$ with $P \in SSpec(N)$. $\mathbb{N}$ denotes the set of positive integers.

Lemma 3.1. *Let N be a near-ring and A be a subset of N.*

(1) *$SSpec(N)$ is a topological space with a base $\{D(a) \mid a \in N\}$.*
(2) *$D(A) = \bigcup_{a \in A} D(a) = D(S(A))$.*
(3) *$\bigcup_{i \in I} D(A_i) = D(\sum_{i \in I} A_i)$ where A_i is a subset of N containing 0 for all $i \in I$.*
(4) *$D(I) \cap D(J) = D(IJ)$ for ideals I, J in N.*
(5) *$D(I) \cap D(J) = \phi$ in $SSpec(N)$ if and only if $IJ \subseteq N^*(N)$ for ideals I, J in N.*
(6) *$S(IJ) = S(I) \cap S(J) \supseteq S(I)S(J)$ for $I, J \in Idl(N)$.*

Proof. Straightforward. □

For any subset A of $SSpec(N)$, we denote $\cap A = \cap_{P \in A} P$.

Lemma 3.2. *Let N be a near-ring. If A is a subset of $SSpec(N)$, then there exists an ideal $J = \cap A$ of N with $cl(A) = V(J)$. In particular, if A is a closed subset of $SSpec(N)$, then $A = V(J)$ for some ideal J of N.*

Proof. Let $P \in V(J)$ and let $D(x)$ be any arbitrary element in the basis $\mathscr{B}$ such that $P \in D(x)$. Suppose that $D(x) \cap A = \phi$. Then $x \in J$, and so $P \in V(x)$, a contradiction. Thus $D(x) \cap A \neq \phi$, and hence, the result follows from Theorem 17.5 of [7]. □

In view of above lemma, we have the following remark.

Remark 3.3. Let N be a near-ring.
(i) The closure of $P \in SSpec(N)$ is $V(P)$.
(ii) A point $P \in SSpec(N)$ is closed if and only if $P \in Max(N)$.
(iii) If $P, Q \in SSpec(N)$ with $cl(P) = cl(Q)$, then $P = Q$.

With the help of Lemma 3.2, we have the following important characterizations of $SSpec(N)$.

Theorem 3.4. *Let N be a near-ring.*

(1) *If F is a closed set and $D(K)$ is an open set in $SSpec(N)$ satisfying $F \cap Max(N) \subseteq D(K)$, then $F \subseteq D(K)$.*
(2) *$SSpec(N)$ is a compact space.*

(3) *Max*(N) *is a compact* T_1*-space.*
(4) *If SSpec*(N) *is normal, then Max*(N) *is a Hausdorff space.*
(5) *If* $N^*(N)=\cap Max(N)$ *and Max*(N) *is a Hausdorff space, then SSpec*(N) *is normal.*

Proof. (1) Suppose that there is $P \in F$ with $P \notin D(K)$. Let $F = V(L)$ for some ideal L of N. Then $K + L \subseteq P$. Hence each maximal ideal M containing P is also contained in F. Thus $M \in F \cap Max(N)$, and so $M \in D(K)$, a contradiction.

(2) Let $\{D(s_j) \mid j \in J\}$ be an open cover of $SSpec(N)$. Hence $SSpec(N) = \cup_{j\in J} D(s_j)$. Then $\phi = \cap_{j\in J}(SSpec(N)\backslash D(s_j)) = \cap_{j\in J}V(s_j) = V(\sum_{j\in J}\langle s_j\rangle)$ which gives $\sum_{j\in J}\langle s_j\rangle = N$. Then there exists $K \subset J$ finite with $1 = \sum_{k\in K} s_k^{'}$, where $s_k^{'} \in \langle s_k\rangle$ which implies $SSpec(N) = \cup_{k\in K} D(s_k^{'})$. Indeed, clearly

$$\cup_{k\in K} D(s_k^{'}) \subseteq SSpec(N)$$

and suppose $P \in SSpec(N)$ with $P \notin \cup_{k\in K} D(s_k^{'})$. Then $s_k^{'} \in P$ for all $k \in K$ which implies $1 \in P$, a contradiction. Hence $SSpec(N)$ is a compact space.

(3) For any $s_i \in N$, $\{D(s_i) \cap Max(N)\}$ is an arbitrary open set of $Max(N)$. Let $\{D(s_i) \cap Max(N) \mid i \in J\}$ be an open cover of $Max(N)$. Hence $Max(N) = (\cup_{i\in J} D(s_i)) \cap Max(N)$. Then $\phi = \cap_{i\in J}(Max(N)\backslash D(s_i)) = (\cap_{i\in J}V(s_i)) \cap Max(N) = V(\sum_{i\in J}\langle s_i\rangle) \cap Max(N)$ which implies $\sum_{i\in J}\langle s_i\rangle = N$. Then there exists $J_1 \subset J$ finite with $1 = \sum_{j\in J_1} s_j^{'}$, where $s_j^{'} \in \langle s_j\rangle$, and so $Max(N) = (\cup_{j\in J_1} D(s_j^{'})) \cap Max(N)$. Therefore $Max(N)$ is a compact space. Let M_1 and M_2 be two distinct elements in $Max(N)$. Then $M_1 \in D(M_2) \cap Max(N)$ and $M_2 \in D(M_1) \cap Max(N)$, and so $Max(N)$ is a T_1-space.

(4) Let M_1 and M_2 be distinct elements in $Max(N)$. Then $\{M_1\}$ and $\{M_2\}$ are closed subsets in both $SSpec(N)$ and $Max(N)$. If $SSpec(N)$ is normal, then there exist disjoint open sets $D(I)$ and $D(J)$ in $SSpec(N)$ such that $\{M_1\} \subseteq D(I)$ and $\{M_2\} \subseteq D(J)$ for some ideals I and J of N, respectively. So $M_1 \in D(I) \cap Max(N)$ and $M_2 \in D(J) \cap Max(N)$, which imply $Max(N)$ is a Hausdorff space.

(5) Let F_1 and F_2 be two disjoint closed subsets of $SSpec(N)$. Then $F_1 \cap Max(N)$ and $F_2 \cap Max(N)$ are also disjoint closed subsets of $Max(N)$. By Theorem 32.3 in [7], $Max(N)$ is normal. So there are open subsets $D(I)$ and $D(J)$ of $SSpec(N)$ such that $F_1 \cap Max(N) \subseteq A$, $F_2 \cap Max(N) \subseteq B$ and $A \cap B = \phi$, where $A = D(I) \cap Max(N)$ and $B = D(J) \cap Max(N)$. Assume $N^*(N) = \cap Max(N)$. Then $IJ \subseteq \cap Max(N) = N^*(N)$ since $D(I) \cap D(J) = D(IJ)$, and so $D(I) \cap D(J) = \phi$. By (1), we have $F_1 \subseteq D(I)$ and $F_2 \subseteq D(J)$. $\square$

Following Sun [10], we define the normality of $Idl(N)$.

Definition 3.5. $Idl(N)$ is called normal if for each pair $I_1, I_2 \in Idl(N)$ with $I_1 + I_2 = N$ there are $J_1, J_2 \in Idl(N)$ such that $I_1 + J_1 = N = I_2 + J_2$ and $J_1 J_2 = 0$.

Definition 3.6. $Max(N)$ is said to be a retract of $SSpec(N)$ if there is a continuous map $f : SSpec(N) \to Max(N)$ such that $f(M) = M$ for each $M \in Max(N)$.

Lemma 3.7. *Let N be a near-ring.*

(1) *$SSpec(N)$ is normal if and only if for each pair $I_1, I_2 \in Idl(N)$ with $I_1 + I_2 = N$ there are $J_1, J_2 \in Idl(N)$ such that $I_1 + J_1 = N = I_2 + J_2$ and $S(J_1)S(J_2) \subseteq N^*(N)$.*
(2) *If $Idl(N)$ is normal, then so is $SSpec(N)$.*
(3) *If $Max(N)$ is a retract of $SSpec(N)$, then N is a weakly pm near-ring.*
(4) *If $Idl(N)$ is normal, then N is a weakly pm near-ring.*

Proof. (1) Suppose that $I_1, I_2 \in Idl(N)$ with $I_1 + I_2 = N$ and let $F_1 = SSpec(N) \backslash D(I_1)$, $F_2 = SSpec(N) \backslash D(I_2)$. Clearly $D(I_1) \cup D(I_2) = D(I_1 + I_2) = D(N) = SSpec(N)$, so F_1 and F_2 are disjoint closed subsets of $SSpec(N)$. If $SSpec(N)$ is normal, then there are disjoint open subsets $D(J_1)$ and $D(J_2)$ of $SSpec(N)$ such that $F_1 \subseteq D(J_1)$ and $F_2 \subseteq D(J_2)$. Since $D(I_1 + J_1) = D(I_1) \cup D(J_1) = SSpec(N)$ and $D(I_2 + J_2) = D(I_2) \cup D(J_2) = SSpec(N)$, we have $I_1 + J_1 = N = I_2 + J_2$. Since $D(J_1)$ and $D(J_2)$ are disjoint, $S(J_1)S(J_2) \subseteq N^*(N)$. Conversely, let F_1 and F_2 be disjoint closed subsets of $SSpec(N)$. Say $F_1 = SSpec(N) \backslash D(I_1)$ and $F_2 = SSpec(N) \backslash D(I_2)$. Since F_1 and F_2 are disjoint, $D(I_1) \cup D(I_2) = SSpec(N) = D(N)$. By Lemma 3.1(3), $I_1 + I_2 = N$. Then there are $J_1, J_2 \in Idl(N)$ such that $I_1 + J_1 = N = I_2 + J_2$ and $S(J_1)S(J_2) \subseteq N^*(N)$. Hence $F_1 \subseteq D(J_1)$ and $F_2 \subseteq D(J_2)$. Clearly $D(J_1) \cap D(J_2) = D(S(J_1)S(J_2))$. By Lemma 3.1(5), $D(J_1)$ and $D(J_2)$ are disjoint.

(2) Straightforward.

(3) Suppose that $P \in SSpec(N)$ and M_1 is any maximal ideal of N containing P. Let $f : SSpec(N) \to Max(N)$ be a continuous retration and $f(P) = M$. Since $\{M\}$ is closed in $Max(N)$, we have $f^{-1}(\{M\})$ is closed in $SSpec(N)$. Since $f^{-1}(\{M\})$ contains the closure of P, $f^{-1}(\{M\})$ also contains M_1. Hence $M_1 = f(M_1) = M$.

(4) Suppose that there is $P \in SSpec(N)$ with $P \subseteq M_1 \cap M_2$ for some distinct M_1, $M_2 \in Max(N)$. Since $Idl(N)$ is normal and $M_1 + M_2 = N$, there are $J_1, J_2 \in Idl(N)$ such that $M_1 + J_1 = N = M_2 + J_2$ and $J_1 J_2 = 0$. Since $J_1 J_2 = 0$, we have $J_1 \subseteq P$ or $J_2 \subseteq P$. If $J_1 \subseteq P$ then $J_1 \subseteq M_1$, a contradiction. The case of $J_2 \subseteq P$ induces a similar contradiction. □

Note that if N is a NI near-ring, then $N^*(N)$ is completely semiprime ideal and $ab \in N^*(N)$ implies $\langle a \rangle \langle b \rangle \subseteq N^*(N)$ for any $a, b \in N$.

Combining Lemma 2.3 and Lemma 3.4, we have the following theorem.

Theorem 3.8. *Let N be a NI and weakly pm near-ring. Then $Max(N)$ is a compact Hausdorff space.*

Proof. By Lemma 3.4(3), $Max(N)$ is a compact space. Let $M_1, M_2 \in Max(N)$ and consider a multiplicative subset

$$S = \{a_1b_1 \cdots a_{n-1}b_{n-1}a_nb_n \mid a_i \notin M_1, b_i \notin M_2, i = 1, 2, \dots, n,\ n \in \mathbb{N}\}.$$

Suppose that $0 \notin S$. Then by Lemma 2.3, there is a strongly prime ideal P of N with $P \cap S = \phi$ and hence $P \subseteq M_1 \cap M_2$, a contradiction. So there exists $a_i \notin M_1$ and $b_i \notin M_2$ such that $a_1b_1 \cdots a_nb_n = 0$. Let $x_1 = \langle a_1\rangle \langle a_2\rangle \cdots \langle a_n\rangle$ and $x_2 = \langle b_1\rangle \langle b_2\rangle \cdots \langle b_n\rangle$ such that $x_1 \notin M_1$ and $x_2 \notin M_2$. Since N is NI, we have $N/N^*(N)$ is reduced. Hence $x_1x_2 \in N^*(N)$. Since $N^*(N)$ is completely semiprime, we have $\langle x_1\rangle \langle x_2\rangle \subseteq N^*(N)$, which implies $(D(x_1) \cap Max(N)) \cap (D(x_2) \cap Max(N)) = \phi$ with $M_1 \in D(x_1) \cap Max(N)$ and $M_2 \in D(x_2) \cap Max(N)$. Therefore $Max(N)$ is a compact Hausdorff space. □

We have the following corollary from Theorem 3.8.

Corollary 3.9 ([3], Lemma 3.4)**.** *If a ring R is NI and weakly pm, then Max(R) is a compact Hausdorff space.*

As an immediate consequence of Theorem 3.8 or Corollary 3.9, we have the following corollary.

Corollary 3.10 ([3], Corollary 3.5)**.** *If R is a 2-primal and pm ring, then Max(R) is a compact Hausdorff space.*

Proposition 3.11. *For a near-ring N the following conditions are equivalent*:
(1) *SSpec(N) is normal.*
(2) *Max(N) is a retract of SSpec(N) and Max(N) is Hausdorff.*

Proof. (1) $\Rightarrow$ (2) Suppose that $SSpec(N)$ is normal. By Theorem 3.4(4), $Max(N)$ is Hausdorff. Without loss of generality we can assume that $N^*(N) = 0$ since $SSpec(N)$ is canonically isomorphic to $SSpec(N/N^*(N))$. Now for each $P \in SSpec(N)$, define $F_P = \{I \in Idl(N) \mid I + P = N\}$. Then F_P has the following properties: (i) if $I_1 + I_2 \in F_P$, then either $I_1 \in F_P$ or $I_2 \in F_P$, (ii) if $I \in F_P$ and $I \subseteq J$, then $J \in F_P$. Let $M_P = \sum\{I \in Idl(N) \mid I \notin F_P\}$. Note that $1 \notin M_P$ and $P \subseteq M_P$. Assume that M_P is not maximal, say $M_P \subset M$ for some maximal ideal M of N. Then $M \in F_P$ and so $M + P = N$ which implies $M = M + M_P \supseteq M + P = N$, a contradiction. Hence M_P is maximal. If P is maximal, then $M_P = P$.

Now we define a mapping $f : SSpec(N) \to Max(N)$ by sending each $P \in SSpec(N)$ to $M_P \in Max(N)$. Let $D(I) \cap Max(N)$ be an arbitrary open subset of $Max(N)$. We claim that, $f^{-1}(D(I) \cap Max(N))$ is an open subset of $SSpec(N)$. Let P be a strongly prime ideal in $SSpec(N)$ such that $P \in f^{-1}(D(I) \cap Max(N))$. Then $f(P) \in D(I) \cap Max(N)$. Therefore $I \nsubseteq f(P)$. Thus $I + P = N$. So there are ideals J_1, J_2 such that $I + J_1 = N = P + J_2$ and $S(J_1)S(J_2) = 0$, which implies $J_2 \nsubseteq P$. Now we show that $D(J_2) \subseteq f^{-1}(D(I) \cap Max(N))$. Let $P_1 \in D(J_2)$. Then $S(J_1) \subseteq P_1$, which gives $I + P_1 = N$. Hence $I \in F_{P_1}$ and $I \nsubseteq f(P_1)$. Then f is continuous.

(2) $\Rightarrow$ (1) Let g be a continuous retraction of $SSpec(N)$ onto $Max(N)$. For a closed subset F of $SSpec(N)$, we have $g(F) = F \cap Max(N)$. If now F_1 and F_2 are disjoint closed subsets of $SSpec(N)$, we can enclose $F_1 \cap Max(N)$ and $F_2 \cap Max(N)$ in disjoint open sets $D(I)$ and $D(J)$ of $Max(N)$, and now $g^{-1}(D(I))$ and $g^{-1}(D(J))$ are open and disjoint in $SSpec(N)$ with $F_1 \subseteq g^{-1}(D(I))$ and $F_2 \subseteq g^{-1}(D(J))$. □

Theorem 3.12. *Let N be a NI near-ring. Then the following conditions are equivalent*:

(1) *N is weakly pm.*

(2) *$SSpec(N)$ is normal.*

(3) *$Max(N)$ is a retract of $SSpec(N)$.*

Proof. (3) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) follows from Lemma 3.7(3) and Proposition 3.11.

(1) $\Rightarrow$ (2) Suppose that N is weakly pm. Then $SSpec(N)$ is normal by Theorem 3.8 and Proposition 3.11 when $Max(N)$ is a retract of $SSpec(N)$. Since N is weakly pm, we can obtain a retraction $f : SSpec(N) \to Max(N)$ by sending each strongly prime ideal to the unique maximal ideal containing it. For a closed subset $\mathbb{F}$ of $Max(N)$, we claim that $f^{-1}(\mathbb{F})$ is closed in $SSpec(N)$. Let $B = \cup\{M \mid M \in \mathbb{F}\}$, $F = \cap\{M \mid M \in \mathbb{F}\}$ and $I = \cap\{P \in SSpec(N) \mid f(P) \in \mathbb{F}\}$.

Let $Q \in SSpec(N)$ with $Q \subseteq B$. Then $Q + F \subseteq B$ clearly, and so there is a maximal ideal M with $Q + F \subseteq M$. Thus we have $M \in \mathbb{F}$ since $\mathbb{F}$ is closed and $F \subseteq M$. Moreover, M is the unique maximal ideal containing Q because N is weakly pm.

Now let $P \in SSpec(N)$ with $I \subseteq P$. Consider any finite subset $\{s_i \mid s_i \notin B,\ i \leq n\}$ where $n \in \mathbb{N}$. Let $t \notin P$. Then $t \notin I$ and so there is $P_1 \in SSpec(N)$ such that $t \notin P_1$ and $f(P_1) \in \mathbb{F}$. Since $s_i \notin B$, we have $s_i \notin P_1$. Hence there exists $z_i, z_j^{'} \in N$ for $i \leq n$, $j \leq n-1$ such that $s_1z_1tz_1^{'}s_2z_2tz_2^{'}\cdots tz_{n-1}^{'}s_nz_nt \notin P_1$. Define a multiplicative subset $X = \{s_1t_1s_2t_2\cdots s_nt_n \mid s_i \notin B,\ t_i \notin P,\ i \leq n,\ n \in \mathbb{N}\}$. Assume $0 \in X$ and say $s_1t_1s_2t_2\cdots s_nt_n = 0$ for some $s_i \notin B$, $t_i \notin P$. Then there are $c_i \in N$, $i \leq n-1$ such that $t = t_1c_1t_2c_2\cdots t_{n-1}c_{n-1}t_n \notin P$. Hence there exists $z_i, z_j^{'} \in N$ for $i \leq n$, $j \leq n-1$ such that

$$s_1z_1tz_1^{'}s_2z_2tz_2^{'}\cdots tz_{n-1}^{'}s_nz_nt \notin I.$$

By Lemma 2.2, $N/N^*(N)$ is symmetric. Since $s_1t_1s_2t_2\cdots s_nt_n = 0$, we have $s_1z_1tz_1^{'}s_2z_2tz_2^{'}\cdots tz_{n-1}^{'}s_nz_nt \in N^*(N)$. Thus $s_1z_1tz_1^{'}s_2z_2tz_2^{'}\cdots tz_{n-1}^{'}s_nz_nt \in P_1$, a contradiction. Then there exists a strongly prime ideal Q of N with $Q \subseteq P \cap B$. Therefore $Q \subseteq P \subseteq M = f(P) = f(Q) \in \mathbb{F}$. Hence $SSpec(N)$ is normal. □

The following is an immediate corollary of Theorem 3.12.

Corollary 3.13 ([3], Theorem 3.7)**.** *Let R be a NI ring. Then the following conditions are equivalent*:

(1) *R is weakly pm.*
(2) *SSpec*(R) *is normal.*
(3) *Max*(R) *is a retract of SSpec*(R).

If N is NI, we obtain the following results.

Theorem 3.14. *Let N be a NI near-ring. Then $N^*(N)_S = \cap V(N^*(N)_S)$ for any subset S of N.*

Proof. Clearly $N^*(N)_S \subseteq \cap V(N^*(N)_S)$. Let $a \in N \backslash N^*(N)_S$. Then $aS \nsubseteq N^*(N)$. Thus $as \notin P$ for some $P \in SSpec(N)$ and $s \in S$. Let $x \in N^*(N)_S$. Then $xS \subseteq N^*(N)$. Since $N^*(N)$ is completely semiprime, we have $\langle x \rangle \langle s \rangle \subseteq N^*(N)$. Since $s \notin P$, we have $x \in P$. Then $N^*(N)_S \subseteq P$. Thus $a \notin P \in V(N^*(N)_S)$ and hence $\cap V(N^*(N)_S) \subseteq N^*(N)_S$. □

Lemma 3.15. *Let N be a NI near-ring and let $a, b \in N$. Then $int\ V(a) \subseteq int\ V(b)$ if and only if $N^*(N)_a \subseteq N^*(N)_b$.*

Proof. Let $int\ V(a) \subseteq int\ V(b)$ for any $a, b \in N$ and let $x \in N^*(N)_a$. Then $xa \in N^*(N)$, and so $\langle x \rangle \langle a \rangle \subseteq N^*(N)$, which implies $SSpec(N) \backslash V(x) \subseteq V(a)$. Then $SSpec(N) \backslash V(x) \subseteq int\ V(a) \subseteq int\ V(b) \subseteq V(b)$, which gives $bx \in N^*(N)$, so $x \in N^*(N)_b$.

Conversely, let $N^*(N)_a \subseteq N^*(N)_b$ and let $P \in int\ V(a)$. Suppose $P \notin V(b)$. Then $b \notin P$. Since $P \in int\ V(a)$, we have $P \notin SSpec(N) \backslash int\ V(a)$. Then by Lemma 3.2, we have $SSpec(N) \backslash int\ V(a) = V(J)$ for some ideal J of N. Since $P \notin V(J)$, we have $c \notin P$ for some $c \in J$, and so $SSpec(N) \backslash int\ V(a) = V(J) \subseteq V(c)$. Clearly $ac \in N^*(N)$ and $bc \notin N^*(N)$. Then $c \in N^*(N)_a$ and $c \notin N^*(N)_b$, a contradiction. Hence $int\ V(a) \subseteq int\ V(b)$. □

Lemma 3.16. *Let N be a NI near-ring. Then for every $a \in N$, $cl(D(a)) = V(N^*(N)_a) = Supp(a) = SSpec(N) \backslash int\ V(a)$.*

Proof. Let $P \in D(a)$ and $x \in N^*(N)_a$ for any $a \in N$. Then $a \notin P$ and $xa \in N^*(N)$. Since $N^*(N)$ is completely semiprime, we have $\langle x \rangle \langle a \rangle \subseteq N^*(N)$, and so $x \in P$. Thus $N^*(N)_a \subseteq P$ and hence $P \in V(N^*(N)_a)$. So $D(a) \subseteq V(N^*(N)_a)$. Let $P_1 \in cl(D(a))$. Then $P_1 \in V(N^*(N)_a)$ since $V(N^*(N)_a)$ is a closed set containing $D(a)$. Let $P \in V(N^*(N)_a)$, and let $D(x)$ be any arbitrary element in the basis $\mathscr{B}$ such that $P \in D(x)$. Suppose $P \notin D(a)$ and suppose $D(x) \cap D(a) = \phi$. Then $D(xa) \subseteq D(x) \cap D(a) = \phi$, and so $xa \in N^*(N)$ which implies $x \in P$, a contradiction. Thus $D(x) \cap D(a) \neq \phi$ and hence $V(N^*(N)_a) \subseteq cl(D(a))$.

If $P \in D(a)$, then $P \in D(a) \cap D(x) \neq \phi$, and so $V(N^*(N)_a) \subseteq cl(D(a))$. Clearly $Supp(a) = V(N^*(N)_a)$. Let $P \in cl(D(a))$ and suppose that $P \in int\ V(a)$. Then there is an open set U of $SSpec(N)$ with $P \in U \subseteq V(a)$, and so $P \notin SSpec(N) \backslash U$, a contradiction. Let $P \in SSpec(N) \backslash int\ V(a)$ and let $D(x)$ be any arbitrary element in the basis $\mathscr{B}$ such that $P \in D(x)$. Suppose that $D(x) \cap D(a) = \phi$. Then $ax \in N^*(N)$, and so $x \in N^*(N)_a$. But $x \notin P$, we have $N^*(N)_a \nsubseteq P$. Hence $P \in D(N^*(N)_a) \subseteq V(a)$, a contradiction. □

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